On phase diagrams for directed percolation problems

1986 J. Phys. A: Math. Gen. 19 L201

(http://iopscience.iop.org/0305-4470/19/4/006)

J. Phys. A: Math. Gen. 19 (1986) L201-L206. Printed in Great Britain

LETTER TO THE EDITOR

On phase diagrams for directed percolation problems†

Anna Chame‡, S L A de Queiroz‡, R R dos Santos‡ and P M Oliveira§

‡ Departamento de Física, PUC/RJ, 22 452 Rio de Janeiro, RJ, Brazil
§ Departamento de Física, UFF, 24 000 Niterói RJ, Brazil

Received 6 November 1985

Abstract. We propose a real space renormalisation group scheme for anisotropic directed bond percolation in two dimensions which, differing from previous phenomenological or small-cell real space renormalisation groups, is sensitive to the existence of an infinite cluster along any lattice direction. This enables us to explore new aspects of the phase diagram for the problem, which were not accessible through existing treatments; in particular, the crossover exponent between one- and two-dimensional behaviour is shown to be exactly one for directed percolation. Numerical estimates are in agreement with known inequalities.

Real space renormalisation group (RSRG) techniques have been extensively used in the study of phase transitions in recent years (for a review see Burkhardt and van Leeuwen (1982)). In particular, small-cell RSRG schemes have consistently proven their reliability as regards the obtaining of overall features of phase diagrams and stability of fixed points in multiparameter spaces. Examples include dilute Ising and Potts magnets (see the review by Stinchcombe (1983)), polymers with and without branches (Family 1980), percolation with first- and second-neighbour bonds (Riera *et al* 1980) and Potts models with first- and second-neighbour interactions (Oliveira *et al* 1984). However, the proper handling of anisotropy in such schemes has shown to be rather tricky: a straightforward extension to 2D anisotropic bond percolation (Chaves *et al* 1979, de Magalhães *et al* 1981) of an RSRG procedure used for the corresponding isotropic problem (see e.g. Oliveira *et al* 1980) gives a totally unstable isotropic fixed point, contrary to what would be expected on physical grounds (still, the exact critical line is obtained in agreement with the result of Sykes and Essam (1963)). Alternative procedures have been proposed which overcome this difficulty (Nakanishi *et al* 1981, Oliveira 1982) and the situation seems to be settled by now.

In this letter we address a similar (but slightly more subtle) question which arises when one allows for directional effects on top of the anisotropic ones. In the context of directed percolation (for a review see Kinzel (1983)), the phase diagram for the anisotropic bond problem on a square lattice has been obtained by Domany and Kinzel (1981) from a phenomenological RG approach (figure 1(a)); the results of Oliveira (1983) from a small-cell RSRG yielded the flow direction along the critical line as well (figure 1(b)). On the other hand, from a perturbative solution of master equations Grassberger (1983) obtained the phase diagram displayed in figure 1(c). Our purposes here are: (i) to show how a convenient RSRG scheme can be set up which reproduces

† Work partially supported by FINEP, CNPq and CAPES.

0305-4470/86/040201+06$02.50

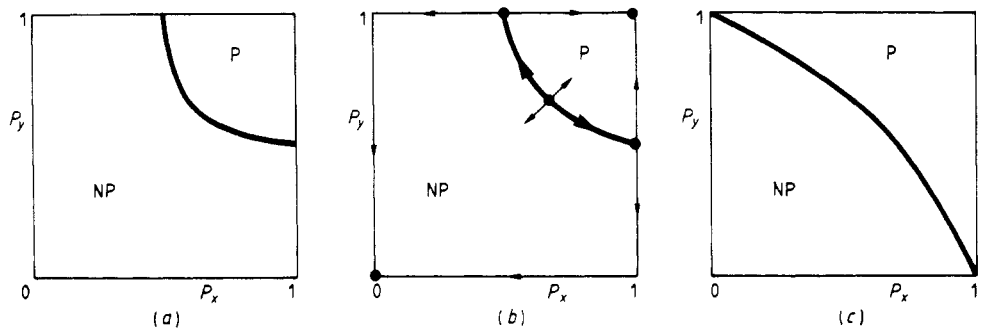

Figure 1. Phase diagrams obtained by (*a*) Domany and Kinzel (1981) (adapted from figure 3 of their paper); (*b*) Oliveira (1983); (*c*) Grassberger (1983) for 2D directed bond percolation on a square lattice. P_x, P_y denote bond probabilities along x and y directions respectively. In (*b*), arrows indicate RG flow directions. P≡'percolating' phase; NP≡'non-percolating' phase.

the features of Grassberger's result; and (ii) to provide further insight into the peculiarities of RSRG when applied to directed problems.

We begin by recalling that the origins of the apparent discrepancy between the diagram of figure 1(*c*) and those of figures 1(*a*) and 1(*b*) are not totally unknown. In the phenomenological RG of Domany and Kinzel, the 'strips' used in renormalisation are infinite along the diagonal of the square lattice, which coincides with the 'easy', or 'time', direction only for the isotropic (that is, $p_x = p_y$) directed problem. Thus, one is always observing percolation along the diagonal; allowing $p_x \neq p_y$ corresponds to rotating the easy direction, or 'rotating the percolating cone' (Domany and Kinzel 1981) in the isotropic problem; in this way one can study critical behaviour at an angle $\phi = \tan^{-1}((p_y/p_x) - \frac{1}{4}\pi)$ to the easy direction, so for sufficiently high anisotropy the percolating cone will fall off the diagonal (even though it may be of infinite extent) and the corresponding point on the phase diagram will be classified by the RG as belonging to the 'non-percolating' region. Accordingly, the RSRG results of Oliveira (1983) are interpreted as meaning that his renormalisation group scheme is sensitive to percolation along the diagonal direction; a similar interpretation also holds for the three-dimensional simple cubic case, to which an extension of this work has been performed (Chame *et al* 1984). Further evidence in favour of this viewpoint comes from the easier problem of directed self-avoiding walks, where it has been shown exactly that an analogous RSRG procedure indeed gives the exponent related to critical behaviour along the 'easy' direction (which in that case coincides with the diagonal) (de Queiroz 1983). As regards the instability, along the critical line, of the isotropic fixed point $p_x = p_y$ (see figure 1(*b*)) Oliveira (1983) shows that this is consistent with the above interpretation of this RSRG and with Domany and Kinzel's results, namely that critical behaviour along a direction other than the 'easy' axis corresponds to the qualitatively distinct process which is the broadening of an already infinite cluster. Note also that both Domany and Kinzel (1981) and Oliveira (1983) are able to study off-diagonal behaviour for the isotropic problem in a few special cases, the former by choosing selected strip directions, the latter by using rectangular ($m \times n$) cells.

On the other hand, the approach of Grassberger (1983) is sensitive to the existence of an infinite cluster along any direction, not just along the diagonal; this is because the quantities sought are the upper and lower angular limits of the percolating cone

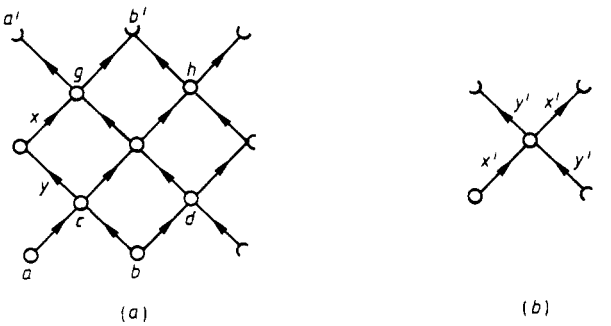

Figure 2. (*a*) Cell used in our RSRG; here the scaling factor $b = 2$. (*b*) Renormalised cell. For x' we count all configurations entering the cell in (*a*) at *a* or *b* and leaving it at *g* or *h*; for y' we count all configurations entering at *c* or *d* and leaving at a' or b'.

($\phi_\pm$), the percolation threshold being defined by $\phi_+ = \phi_-$. Owing to the nature of his approach, it is not possible to extract information on critical exponents or on the crossovers between different universality classes from the data; this is why an RSRG formulation will play a complementary role to that of Grassberger, provided that it takes proper account of suitable geometric features of the problem. Our task here is to propose such a formulation, and to discuss the physical picture that emerges from it. We have made use of cells such as the one shown in figure 2(*a*) for a scaling factor $b = 2$; apart from the inclusion of bond directionality (and the respective connective constraints), these are the same cells already used for the study of undirected anisotropic bond percolation (Oliveira 1982). We have performed RSRG transformations for scaling factors $b = 2$ and $b = 3$; the overall features of the phase diagram thus obtained are displayed in figure 3. In addition to the trivial fixed points at $(x, y) = (0, 0)$ and $(1, 1)$ we obtain a pair of non-trivial ones, describing one-dimensional behaviour, at A(0, 1) and B(1, 0); finally the isotropic point $P(x_p = y_p)$ governs the behaviour along the critical line ABP. The shape of the phase diagram is then the same as Grassberger's, which means that our RSRG senses an infinite cluster along any direction; this can be

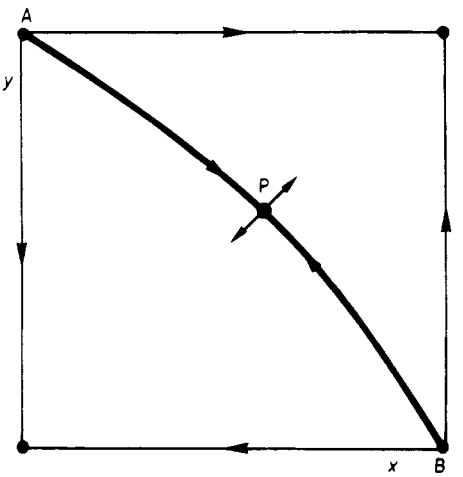

Figure 3. Phase diagram (schematic) obtained from the RSRG transformation depicted in figure 2.

qualitatively understood from the fact that, since we allow any entry point at the bottom of the cell (see figure 2) and because of its specific shape, connectivity along any direction is accounted for. This is in contrast to the scheme of Oliveira (1983) where there is only *one* entry, at the bottom-left corner of the cell: there, the statistically dominant configurations will be those that leave the cell close to the diagonal.

Also, the stability of P along ABP means that no matter how far the percolating cluster is rotated from the diagonal, it will have the same asymptotic behaviour (provided that it does not turn one-dimensional; this leaves A and B in a different universality class, as it should be).

Before turning to the detailed description of the critical behaviour observed at P we point out an exact result obtained from our RSRG, regarding the crossover between one- and two-dimensional behaviour. At both A(0, 1) and B(1, 0), it is easy to show (by exactly enumerating the paths that contribute to connectivity) that, for any scaling factor b one has $\partial x'/\partial x = \partial y'/\partial y = 2b-1$ and $\partial x'/\partial y = \partial y'/\partial x = 0$. Hence the Jacobian matrix is diagonal and has identical eigenvalues ($\lambda_1 = \lambda_2 = 2b-1$) at both A and B; this means that the crossover exponent ϕ (given by $\ln \lambda_2/\ln \lambda_1$) is exactly *one*. Although an identical result has been derived for *undirected* anisotropic percolation along the same lines of reasoning (Oliveira 1982), to our knowledge it has not been obtained previously for *directed* problems. In this connection it is worth noting that, since in one dimension one still has the mean-cluster size exponent $\gamma = 1$ (Reynolds *et al* 1977) in the directed problem, our results means that $\phi_{d,d-1} = \gamma_{d-1}$ for directed percolation at least in $d = 2$. Whether this still holds true for directed percolation in $d > 2$ (thus generalising the argument of Redner and Coniglio (1980) regarding anisotropic *undirected* percolation) is an interesting question which, however, we shall not pursue here. Further, we note that in our case $\phi = 1$ has the meaning that the critical line leaves the one-dimensional points A and B at a *finite* angle with the coordinate axes; this is then an exact result, which is already shown in the approximate phase diagram of Grassberger (see figure 1(*c*)). Finally, since the critical exponent estimate is given by $\nu = \ln b/\ln \lambda$, and with $\lambda = 2b-1$, we see that as $b \to \infty$ one has $\nu \to 1$ at both A and B, which is the exact value of ν for 1D (directed or undirected) percolation (Reynolds *et al* 1977).

Numerical results obtained for the location of the isotropic fixed point P, and corresponding eigenvalues, are displayed in table 1. The calculated estimates for the critical probability are consistently higher than $\frac{1}{2}$ (the exact threshold for the undirected problem) as it should be, since the spread of connectivity is hampered by directional

Table 1. Location of the isotropic fixed point P, eigenvalues of the linearised RG transformation around P and critical exponent estimate $\nu = \ln b/\ln \lambda_1$.

	$b=2$	$b=3$	Accepted values
$x_p = y_p$	0.5021	0.5145	0.644 ± 0.001[a]
λ_1	1.9414	2.6604	
λ_2	0.3164	0.1757	
ν_p	1.0448	1.1227	1.734 ± 0.002[b]

[a] Kinzel and Yeomans (1981).
[b] Kinzel (1983).

constraints; that they still are somewhat far off the accepted values of ~ 0.64 is attributable to the essentially uncontrolled nature of the approximations involved in small-cell RSRG calculations. It must be noticed, however, that the situation improves somewhat from $b=2$ to $b=3$; a naive two-point extrapolation based on finite-size arguments (Fisher 1971) gives $p_c(\text{ext}) \simeq 0.56$. The results of Oliveira (1983) behave differently: although closer to the accepted value (varying between 0.671 and 0.684 as b goes from 2 to 4), they seem to be moving slightly *away* from it as b grows; it is believed that this trend must be related to the type of approximations referred to above, and should be reversed for larger scaling factors.

As regards the correlation length exponent calculated at P, it is to be compared with $\nu_\parallel$, which describes the divergence of the percolating cluster along the 'easy' direction and, in two dimensions, has the value ~ 1.73 quoted in table 1 (see e.g. Klein and Kinzel (1981) for a discussion of the meaning of $\nu_\parallel$ and $\nu_\perp$, this latter describing the spread of indirect correlations along directions perpendicular to the easy axis). This can be seen more easily by invoking the similarity between our phase diagram and that of Grassberger (1983), and recalling his definition of the percolation threshold as being marked by the existence of a zero-width infinite cone; the direction along which the cone first becomes infinite is the 'easy' axis (see the discussions in Domany and Kinzel (1981) and Oliveira (1983)). Thus its length diverges as $\Delta p^{-\nu_\parallel}$, where Δp has the meaning of a distance (in two-dimensional parameter space) from the critical boundary. Note also that a third exponent (called $\nu(\phi)$ by Domany and Kinzel (1981), believed to be equal to 2 for any $\phi \neq 0$) is absent from our considerations. This is bound to be so, for $\nu(\phi)$ describes the divergence of the correlation length along directions at an angle ϕ with the easy axis, so it can only be detected by means of 'rotating the percolating cone' in schemes such as the strips of Domany and Kinzel (1981) or the cells of Oliveira (1983) which are sensitive to percolation along a fixed direction in space. This is clearly not our case, as already stated above.

Our estimates for $\nu_\parallel$ are somewhat smaller than the accepted value of ~ 1.73 (see table 1); however, they tend to be larger (and the difference increases from $b=2$ to $b=3$) than their counterparts obtained by using the same cells for undirected bond percolation, namely $\nu(b=2)=1.042$ and $\nu(b=3)=1.099$ (Riera *et al* 1980, Oliveira 1982), as it should be. A simple two-point extrapolation assuming $1/\nu(b) - 1/\nu(\text{true}) \sim 1/\ln b$ (Reynolds *et al* 1978, 1980) gives $\nu(\text{true}) \sim 1.29$; of course, this must be taken into account only as showing a qualitative trend of the exponent, not as having a precise numerical meaning. We note that the results of Oliveira (1983) increase from 1.430 to 1.591 as b varies from 2 to 4; a similar extrapolation of his values gives $\nu(\text{true}) \sim 1.77$.

In summary, we have proposed an RSRG scheme for two-dimensional anisotropic directed bond percolation which, differing from previous phenomenological or small-cell RSRG formulations, is sensitive to the existence of an infinite cluster along any direction on the lattice. This enables us to discuss aspects of the phase diagram for the problem which are not accessible through existing treatments; the extreme anisotropic (one-dimensional) limit has been given special attention, and the crossover exponent between one- and two-dimensional behaviour has been shown to be exactly one. Although our numerical results at the isotropic fixed point are not particularly accurate, they all obey known inequalities.

We would like to thank P Grassberger for having drawn our attention to this problem.

References

Burkhardt Th and van Leeuwen J M J (ed) 1982 *Real Space Renormalisation* (Berlin: Springer)
Chame A, de Queiroz S L A and dos Santos R R 1984 *J. Phys. A: Math. Gen.* **17** L657
Chaves C M, Oliveira P M, de Queiroz S L A and Riera R 1979 *Prog. Theor. Phys.* **62** 1550
de Magalhães A C N, Tsallis C and Schwachheim G 1981 *J. Phys. C: Solid State Phys* **14** 1393
de Queiroz S L A 1983 *J. Phys. A: Math. Gen.* **16** L617
Domany E and Kinzel W 1981 *Phys. Rev. Lett.* **47** 5
Family F 1980 *J. Phys. A: Math. Gen.* **13** L325
Fisher M E 1971 *Critical Phenomena: Enrico Fermi Summer School Varenna, Italy* Course 51, ed M S Green (New York: Academic) p 1
Grassberger P 1983 *J. Phys. A: Math. Gen.* **16** 591
Kinzel W 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 425
Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen.* **14** L163
Klein W and Kinzel W 1981 *J. Phys. A: Math. Gen.* **14** L405
Nakanishi H, Reynolds P J and Redner J 1981 *J. Phys. A: Math. Gen.* **14** 855
Oliveira P M 1982 *Phys. Rev.* B **25** 2034
—— 1983 *J. Physique Lett.* **44** 495
Oliveira P M, de Queiroz S L A, Riera R and Chaves C M 1980 *J. Phys. A: Math. Gen.* **13** 2457
Oliveira P M, Tsallis C and Schwachheim G 1984 *Phys. Rev.* B **29** 2755
Redner S and Coniglio A 1980 *Phys. Lett.* **79A** 111
Reynolds P J, Stanley H E and Klein W 1977 *J. Phys. A: Math. Gen.* **10** L203
—— 1978 *J. Phys. A: Math. Gen.* **11** L199
—— 1980 *Phys. Rev.* B **21** 1223
Riera R, Oliveira P M, Chaves C M and de Queiroz S L A 1980 *Phys. Rev.* B **22** 3481
Stinchcombe R B 1983 *Phase Transitions and Critical Phenomena* vol 7, ed C Domb and J L Lebowitz (New York: Academic) p 151
Sykes M F and Essam J W 1963 *Phys. Rev. Lett.* **10** 3